$\mathrm{col}(A) = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_n)$

Note 5: Let A be a $m \times n$ matrix. By theorem 3 from lecture 10, col(A) is a *subspace* of $\mathbb{R}^m$.

Example 3: Let $A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$. Find a basis for col($A$). Calculate dim(col($A$)) and rank($A$). What do you observe?

$B_1 = \{\vec{v}_1\}$

$B_2 = \{\vec{v}_1, \vec{v}_2\}$

$B_3 = \{\vec{v}_1, \vec{v}_2\}$ $\quad \vec{v}_3 = \vec{v}_2$

$\beta = B_4 = \{\vec{v}_1, \vec{v}_2\}$ $\quad \vec{v}_4 = \vec{v}_1$

$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ basis for col(A)

$\dim(A) = 2$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2 := R_2 - R_1]{R_3 := R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\mathrm{rank}(A) = 2$

observation: $\dim(A) = \mathrm{rank}(A)$

Example 4: Let $A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 2 & 3 & 1 \\ 2 & -3 & 6 & 9 & 4 \\ 3 & -1 & 2 & 4 & 1 \\ 7 & -2 & 4 & 8 & 1 \end{bmatrix} \end{array}$ with reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $= C$

Find a basis for col(A). Calculate dim(col(A)) and rank(A). What do you observe?

$B_1 = \{\vec{v}_1\}$

$B_2 = \{\vec{v}_1, \vec{v}_2\}$

$B_3 = \{\vec{v}_1, \vec{v}_2\}$ $\quad \vec{v}_3 = -2\vec{v}_2$

$B_4 = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

$\beta = B_5 = \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ $\quad \vec{v}_5 = (-1)\vec{v}_1 + (4)\vec{v}_2 + 0\vec{v}_3 + 2\vec{v}_4$

$\begin{bmatrix} -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (4)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix} + 4\begin{bmatrix} -1 \\ -3 \\ -1 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 9 \\ 4 \\ 8 \end{bmatrix}$

$\dim(\mathrm{col}(A)) = 3$

$\mathrm{rank}(A) = 3$ ✓

Theorem 1: If A is a $m \times n$ matrix, then

$$\mathrm{rank}(A) = \dim(\mathrm{col}(A)) \tag{2}$$